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# ON THE STABILITY OF THE NATURAL UNSTRESSED STATE OF VISCOELASTIC BODIES

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Within the framework of the Cauchy problem, a class of models of a linear viscoelastic body subjected to the stability principle of the natural unstressed state state of viscoelastic bodies (Principle Y) is isolated in [1]. The principle Y is formulated as follows. Let the boundary conditions be such that the appropriate elasticity theory problem has a zero solution. If a viscoelastic body is free of external loads at each instant $t > 0$, then for every initial state, strain of the body vanishes as $t \to \infty$. The principle Y is called partial if it is satisfied only for some particular class of viscoelasticity problems.

Sufficient conditions for compliance with the partial Y principle are obtained in this paper for models of viscoelastic bodies within the framework of the fundamental initial-boundary value problems for finite bodies.

**1. Formulation of the problem.** The law connecting the strain $\varepsilon_{kl}$ and stress $\sigma_{kl}$ tensors is taken [1, 2] as

$$C\ (a_t)\ \sigma_{kl} = A\ (\partial_t)\ \varepsilon_{rr}\delta_{kl} + 2B\ (\partial_t)\ \varepsilon_{kl}, \quad 2\varepsilon_{kl} = a_{k,l} + a_{l,k}$$

Here $A\ (p),\ B\ (p),\ C\ (p)$ are polynomials of degree $m_A,\ m_B,\ m_C$ respectively, $m_A \leqslant m_B,\ m_C \leqslant m_B$, the symbol $\delta_{kl}$ is the Kronecker delta, $\partial_t$ is the partial dedivative with respect to the time $t$, $a_k$, (where $k = 1,\ 2,\ 3$) are the components of the displacement vector $\mathbf{a}$, the subscript $(l)$ after the comma denotes differentiation with respect to the corresponding space coordinate $(x_l)$, and summation is taken over repeated subscripts.

The linear viscoelasticity equations are

$$B\ (\partial_t)\ \Delta\mathbf{a} + [A\ (\partial_t) + B\ (\partial_t)]\ \mathrm{grad\ div}\ \mathbf{a} - C\ (\partial_t)\ \rho\partial_t^2\ \mathbf{a} = -C\ (\partial_t)\ \mathbf{F} \quad (1.1)$$

The first initial-boundary value problem of linear viscoelasticity, the Problem A is considered (see Notes 3.1 and 3.2 for the remaining problems)

$$\partial_t^k\ \mathbf{a}\ |_{t=0} = \mathbf{b}_k\ (0 \leqslant k \leqslant N \equiv \max\ \{m_B - 1,\ \ m_C + 1\}), \qquad (1.2)$$
$$\mathbf{a}\ |_\Gamma = 0$$

Here $\rho > 0$ is the density of the material, $\mathbf{F}$ are the external body forces, $\Gamma$ is the boundary of the finite volume $\Omega$ which is henceforth assumed sufficiently smooth.

The solution of the Problem A is sought in the form

$$\mathbf{a} = \mathbf{u} + \mathbf{u}_0, \quad \mathbf{u}_0\,(\mathbf{x},\ t) = \chi\,(t)\sum_{k=0}^{N} t^k\mathbf{b}_k$$

where $\chi\ (t)$ is a fixed, infinitely differentiable function, equal to unity in the neighborhood of the point $t = 0$ and to zero for $t > 1$. The vector function $\mathbf{u}$ evidently satisfies the homogeneous boundary and initial conditions (1.2) and is a solution of (1.1) with a known altered right side $\mathbf{\Phi}$.

The Laplace transform

$$\mathbf{v}\ (x,\ p) = L\mathbf{u} \equiv \int_0^\infty e^{-pt}\ \mathbf{u}\,(\mathbf{x},\ t)\,dt$$

is used to investigate Problem A, and results in some boundary value problem with a parameter, i.e. Problem B. Problem B is posed in a generalized formulation below (Definition 3.1).

**2. Auxiliary material.** The following spaces are introduced. Let $H$ be some separable, complex Hilbert space.

The space $E_k\ (\gamma,\ H)$ is a space of functions $\varphi\ (p)$ with values in $H$ which are analytic ( [3], p. 184) in the half-plane $\mathrm{Re}p \equiv \sigma > \gamma \geqslant 0$ and have the finite norm

$$\|\varphi\|^2_{E_k(\gamma,\,H)} = \sup_{\sigma>\gamma}\int_{-\infty}^{\infty}\|\varphi\,(\sigma + i\tau)\|^2_H\,(1 + |\sigma + i\tau\,|^{2k})d\tau$$

The space $P_k\ (\gamma,\ H)\ (k \geqslant 0$ is an integer and $\gamma \geqslant 0)$ is a complex space of functions $f\ (t)$ with values in $H$, possessing generalized derivatives $\partial_t^r f$ (see [4]) up to order $k$ with respect to $t$ inclusive on $R_+ = [0,\ \infty)$ such that $\partial_t^r f = 0$ for $t = 0\ (0 \leqslant r < k)$ and the finite norm is

$$\|f\|^2_{P_k(\gamma, H)} = \int_0^\infty \sum_{r=0}^k e^{-2\gamma t} \|\partial_t^r f\|^2_H \, dt$$

Analogous spaces for the scalar functions have been introduced in [5].

The space $L^2(R_+, S)$ [4] which is the space of functions $q(t)$ with values in a Banach space $S$ having the finite norm

$$\|q\|^2_{L^2(R_+, S)} = \int_0^\infty \|q(t)\|^2_S \, dt$$

will also be used later.

The following theorem holds.

Theorem P.—W. The Laplace transform operator $L$ continuously maps the space $P_k(\gamma, H)$ ($k \geqslant 0$ is an integer, $\gamma \geqslant 0$) onto the space $E_k(\gamma, H)$. The operator $L$ is continuously invertible and its inverse is the inverse Laplace transform operator. Namely, if $\varphi(p) \in E_k(\gamma, H)$, then

a) there exists a function $\varphi(\gamma + i\tau)$ such that

$$\lim_{\sigma \to +\gamma} \int_{-\infty}^{\infty} \|\varphi(\gamma + i\tau) - \varphi(\sigma + i\tau)\|^2_H \, d\tau = 0$$

b) there exists a function $f(t) \in P_k(\gamma, H)$, such that

$$\lim_{M \to \infty} \int_0^\infty e^{-2\gamma t} \sum_{r=0}^k \left\| \partial_t^r f(t) - \int_{\gamma - iM}^{\gamma + iM} (i\tau)^r \varphi(\gamma + i\tau) e^{i\tau t} \, d\tau \right\|^2_H dt = 0$$

and $\varphi(p) = Lf(t)$

The Theorem P.—W. is a generalization of the Paley-Wiener theorem (see [5]). To prove its first part, an arbitrary element $f(t)$ from $P_k(\gamma, H)$ is expanded in a series in the complete orthonormalized basis $\psi_n$ of the space $H$

$$f(t) = \sum_{n=1}^\infty d_n(t) \psi_n. \quad d_n(t) = (f \cdot \psi_n)_H$$

From the form of the norm

$$\|f\|^2_{P_k(\gamma, H)} = \int_0^\infty \sum_{n=1}^\infty e^{-2\gamma t} \sum_{r=0}^k |\partial_t^r d_n(t)|^2 \, dt$$

and the definition of the space $P_k(\gamma, H)$ there results that $d_n(t) \in P_k(e^{-\gamma t})$ $\gamma > 0$ (see [5]). From Theorem 7.1. in [5] (see also the remark there), it follows that $\alpha_n = Ld_n \in E_k(\gamma)$ (see [5]), and the operator $L$ is continuous. Hence, and from the orthonormality of the basis $\psi_n$ there results that

$$\varphi = Lf \equiv \sum_{n=1}^\infty \alpha_n \psi_n$$

belongs to the space $E_k(\gamma, H)$ and the operator $L$ acts continuously from $P_k(\gamma, H)$ into $E_k(\gamma, H)$. The reverse assertion of the P.—W. Theorem is proved analogously.

Later, estimates of the polynomials $P(p, \alpha, \beta)$ with the real coefficients

$$P(p, \alpha, \beta) = D(p, \alpha) + \beta p^2 C(p)$$
$$D(p, \alpha) = d_0(\alpha) p^n + \ldots + d_n(\alpha), \quad C(p) = c_0 p^q + \ldots + c_q, \quad c_0 > 0$$

will be obtained, where $d_k(\alpha)$ are continuous functions of the parameter $\alpha$, $q \equiv m_C \leqslant n$, $n > 0$.

Theorem 2.1. For all $\alpha \in [0, \alpha^\circ]$, $\beta \in [0, \beta^\circ]$; $\alpha^\circ, \beta^\circ < \infty$, let all roots $p_k$ of the polynomial $P(p, \alpha, \beta)$ lie in the domain $\operatorname{Re} p \equiv \sigma < 0$ and $d_0(\alpha) \neq 0$. Then the following estimate holds

$$|P(p, \alpha, \beta)| \geqslant m_1 (1 + |p|^r), \quad \operatorname{Re} p \geqslant 0 \tag{2.1}$$

$$r = \begin{cases} n, & m_C + 1 \leqslant n \\ n - 1, & m_C = n \end{cases}$$

$$\inf \{c_1 d_0(\alpha) - c_0 d_1(\alpha)\} = m_2 > 0, \quad \alpha \in [0, \alpha^0]$$

Here and henceforth $m_k > 0$ are certain positive constants.

Proof. Expansion of the polynomial $P(p, \alpha, \beta)$ into prime factors and the condition $\operatorname{Re} p_k < 0$ show that the estimate (2.1) holds if the coefficient of the highest power in $p$ does not vanish. This latter is always valid for $n \geqslant q + 2$, and if $\beta \in [\delta, \beta^\circ]$, where $\delta > 0$ is any number, then also for $n \leqslant q + 1$. (It must be taken into account that the set of roots $p_k$ forms a bounded closed set in the domain $\sigma < 0$ on the plane of the complex variable $p = \sigma + i\tau$ for the mentioned $\alpha, \beta$.) In particular, the estimate (2.1) holds for the polynomial $D(p, \alpha)$. Hence, it follows that for every $M < \infty$, so small a $\delta_0 > 0$ can always exist such that the estimate (2.1) will also hold for $|p| \leqslant M$ and $\beta \in [0, \delta_0]$.

There remains to prove the estimate (2.1) under the conditions $P$: $|p| \geqslant M$, $\sigma_0 > 0$, $\alpha \in [0, \alpha^0]$, $\beta \in [0, \delta_0]$, $n \leqslant q + 1$, where $M$ is sufficiently large.

Let $n = q + 1$. Since all the coefficients of the polynomial $P(p, \alpha, \beta)$ are positive (this follows from the condition $\operatorname{Re} p_k < 0$), we deduce

$$|P(p, \alpha, \beta)| = |p|^n |d_0(\alpha) + \beta c_0 \sigma + \beta c_1 + i\tau\beta c_0 + p^{-1} P_1(p^{-1}, \alpha, \beta)| \geqslant |p|^n [d_0(\alpha) - m_3 M^{-1}] \geqslant m_4 |p|^n > 0$$

$$m_3 = \sup |P_1(p^{-1}, \alpha, \beta)| < \infty \quad \text{for } \alpha \in [0, \alpha^\circ], \quad \beta \in [0, \beta^\circ], \ |p| \geqslant 1$$

Now let $n = q$. Then

$$|P(p, \alpha, \beta)| = |p|^{n-1} |\chi_1(p, \alpha, \beta) + i\chi_2(p, \alpha, \beta) + p^{-1} P_2(p^{-1}, \alpha, \beta)|$$

$$\chi_1(p, \alpha, \beta) = -\beta(3c_0\sigma + c_1)\tau^2 + \beta(c_0\sigma^3 + c_1\sigma^2 + c_2\sigma + c_3) + d_0\sigma + d_1 \ \chi_2(p, \alpha, \beta) = -\beta c_0 \tau^3 + \beta\tau(3c_0\sigma^2 + 2c_1\sigma + c_2) + d_0(\alpha)\tau$$

(if $q < 3$, then $c_s = 0$ for $s > q$). It is seen that

$$|P_2(p^{-1}, \alpha, \beta)| \leqslant m_5 < \infty, \quad \text{for } \beta \in [0, \beta^\circ], \quad |p| \geqslant 1, \quad \alpha \in [0, \alpha^\circ]$$

If $|\chi_1| \geqslant \eta > 0$, then under the conditions $P$ the estimate (2.1) holds with the constant $m_1 = {}^1/_2 \eta$. In particular, this is valid if $|\tau| \leqslant {}^1/_2 \sigma$ and $\delta_0$ is sufficiently small.

Let $|\chi_1| \leqslant \eta$ and $|\tau| \geqslant {}^1/_2\sigma$, then

$$|\chi_2(p, \alpha, \beta)| = |\tau(3c_0\sigma + c_1)^{-1}| \, |c_1 d_0 - c_0 d_1 + \beta(8c_0^2\sigma^3 + 8c_0c_1\sigma^2 + 2c_0c_2\sigma + 2c_1^2\sigma + c_1c_2 - c_0c_3) + (c_0\chi_1 + 2c_0 d_0 \sigma)|$$

If $M$ is sufficiently large, and $\delta_0$, $\eta$ are small, then the value of the second factor in the right side can be given a lower bound in terms of $^1/_2 m_2$. Indeed, a number $\sigma_0 > 0$ can be indicated such that for $\sigma \geqslant \sigma_0$ all the terms of this factor in parentheses are positive. If $0 \leqslant \sigma \leqslant \sigma_0$, then this estimate holds for sufficiently small $\delta_0 > 0$. The estimate (2. 1) hence results.

It must be noted that an additional condition in case $r = n - 1$ for Theorem 2. 1 will reinforce the main condition $\operatorname{Re} p_k < 0$, since the first Hurwitz determinant for the polynomial $P(p, \alpha, \beta)$, which should be positive, is

$$\beta\,[c_1 d_0(\alpha) - c_0 d_1(\alpha)] + \beta^2 (c_1 c_2 - c_0 c_3) > 0$$

**3. Sufficient conditions for compliance with the partial principle Y.** Let $\mathbf{H}$ be a complex space of vector-functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ formed by the closure of functions $\varphi_k$ continuously differentiable in $\Omega$ and equal to zero on $\Gamma$ in the norm generated by the scalar product

$$(\varphi\cdot\psi)_{\mathbf{H}} = \int_\Omega \varphi_{k,l}\overline{\psi}_{k,l}\, d\Omega$$

where $\overline{\psi}_k$ is the complex-conjugate to the function $\psi_k$. Evidently

$$\mathbf{H} = W_2^{\circ(1)}(\Omega) \times W_2^{\circ(1)}(\Omega) \times W_2^{\circ(1)}(\Omega)$$

Definition 3. 1. The vector function $\mathbf{v} \in \mathbf{H}$ satisfying the equality

$$\int_\Omega \{B(p)\, v_{k,l}\overline{\varphi}_{k,l} + [A(p) + B(p)] \operatorname{div}\mathbf{v} \operatorname{div}\overline{\varphi}\}\, d\Omega + \qquad (3.1)$$

$$\int_\Omega p^2 C(p)\, \rho v_k \overline{\varphi}_k\, d\Omega = \int_\Omega f_k \overline{\varphi}_k\, d\Omega, \quad \mathbf{f} \equiv (f_1, f_2, f_3) = L\Phi$$

for any vector function $\varphi \in \mathbf{H}$ is called a generalized solution of the Problem B for a fixed $p$. For the definition to be correct it is required that the right side of (3. 1) be a continuous functional in $\varphi$ in the space $\mathbf{H}$.

Using the Riesz theorem about the representations of a continuous linear functional in Hilbert space, (3. 1) can be written as an operator equation in the space $\mathbf{H}$

$$B(p)\mathbf{v} + [A(p) + B(p)]\,\mathbf{G}_1\mathbf{v} + \rho p^2 C(p)\,\mathbf{G}_2\mathbf{v} = \mathbf{Kf} \qquad (3.2)$$

$$(\mathbf{G}_1\mathbf{v}\cdot\varphi)_{\mathbf{H}} = \int_\Omega \operatorname{div}\mathbf{v} \operatorname{div}\overline{\varphi}\, d\Omega$$

$$(\mathbf{G}_2\mathbf{v}\cdot\varphi)_{\mathbf{H}} = \int_\Omega v_k\overline{\varphi}_k\, d\Omega, \quad (\mathbf{Kf}\cdot\varphi)_{\mathbf{H}} = \int_\Omega f_k\overline{\varphi}_k\, d\Omega$$

Lemma 3. 1. The operators $\mathbf{G}_1$ and $\mathbf{G}_2$ are continuous and positive in the space $\mathbf{H}$. The operator $\mathbf{K}$ acts continuously from the space $W_2^{\circ(-1)}(\Omega) \times W_2^{\circ(-1)}(\Omega) \times W_2^{\circ(-1)}(\Omega)$ in $\mathbf{H}$; $\|\mathbf{G}_1\| = 1$.

Corresponding properties of the operators result from their definition and the imbedding theorem of Sobolev [6]. In [7] it is proved that $\|\mathbf{G}_1\| = 1$. The operator of the left side of (3. 2), denoted by $\mathbf{T}(p)$ is considered. The following identity holds:

$$(\mathbf{T}(p)\,\varphi\cdot\varphi)_{\mathbf{H}} = \|\varphi\|_{\mathbf{H}}^2 P_0(p, \alpha, \beta) \qquad (3.3)$$

$$P_0(p, \alpha, \beta) = B(p) + \alpha\,[A(p) + B(p)] + \beta p^2 C(p)$$

$$\alpha = \|\varphi\|_{\mathbf{H}}^{-2} (\mathbf{G}_1\varphi\cdot\varphi)_{\mathbf{H}}, \quad \beta = \rho\|\varphi\|_{\mathbf{H}}^{-2} (\mathbf{G}_2\varphi\cdot\varphi)_{\mathbf{H}}$$

There follows from Lemma 3.1 that $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \beta_0 < \infty$.

Lemma 3.2. Let the polynomial $P_0(p, \alpha, \beta)$ satisfy all the conditions of Theorem 2.1, where $\alpha^\circ = 1$, $\beta^\circ = \beta_0$, and $\mathbf{Kf} \in E_0(0, \mathbf{H})$. Then (3.2) is uniquely solvable in $\mathbf{H}$ for all $\operatorname{Re} p \geqslant 0$, and its solution is $\mathbf{v}(p) \in E_k(0, \mathbf{H}) \cap E_r(0, \mathbf{L}^2(\Omega))$, where $k = r = n$ for $m_C + 1 \geqslant n$ and $k + 1 = r = n$ for $m_C = n$.

There follows from the identity (3.3) and Theorem 2.1 that for $\operatorname{Re} p \geqslant 0$ the operator $\mathbf{T}(p)$ possesses a continuous inverse operator $\mathbf{T}^{-1}(p)$. It is seen that the operator $\mathbf{T}'(p)$, the conjugate of the operator $\mathbf{T}(p)$ is $\mathbf{T}(p)$. There results from the above that $\mathbf{T}'(p)\varphi = 0$ for $\operatorname{Re} p \geqslant 0$, if and only if $\varphi = 0$. According to the Banach theorem on operators with a closed domain of values ([3], p. 284), the domain of values of the operator $\mathbf{T}(p)$ for $\operatorname{Re} p \geqslant 0$ is the whole space $\mathbf{H}$. Therefore, (3.2) is uniquely solvable for all $\mathbf{Kf} \in \mathbf{H}$. Furthermore, $\mathbf{T}(p)$ is an entire operator-function of the parameter $p$. Hence, and from the continuity of the operator $\mathbf{T}^{-1}(p)$ there results the analyticity of the solution $\mathbf{v}(p)$ of (3.2) in the domain $\operatorname{Re} p > 0$, where the function $\mathbf{Kf}(p)$ is analytic.

To obtain an estimate of the solution $\mathbf{v}(p)$, Eq. (3.2) in which its solution has been substituted, is multiplied scalarly term-by-term in the space $\mathbf{H}$ by $\mathbf{v}(p)$. From the obtained equality, and taking account of (3.3)' and Theorem 2.1, the following estimate is deduced (for $\gamma = 0$):

$$\|\mathbf{v}(p)\|_{\mathbf{H}}^2 (1 + |p|^{2k}) + \gamma \|\mathbf{v}(p)\|_{\mathbf{L}^2(\Omega)}^2 (1 + |p|^{k+q+2}) \leqslant m_6 \|\mathbf{Kf}\|_{\mathbf{H}}^2$$

where $k$ is defined by the conditions of the lemma, $\operatorname{Re} p \geqslant 0$, $m_6 < \infty$. Taking into account this estimate, the definition of the operator $\mathbf{G}_2$ and returning to the equality obtained above, an estimate can be deduced for $\gamma = 1$ also (with an altered constant $m_6$), which completes the proof of the lemma.

Application of the P.—W. theorem now assures the existence of a unique generalized solution $\mathbf{u}(t)$ of the Problem A in the following sense: $\mathbf{u}(t)$ satisfies the equation

$$\int_0^\infty \{(\mathbf{u}\cdot B(-\partial_t)\psi)_{\mathbf{H}} + (\mathbf{G}_1\mathbf{u}\cdot[A(-\partial_t) + B(-\partial_t)]\psi)_{\mathbf{H}} + (\mathbf{G}_2\mathbf{u}\cdot\rho\partial_t^2 C(-\partial_t)\psi)_{\mathbf{H}} - (\mathbf{K\Phi}\cdot\psi)_{\mathbf{H}}\}\, dt = 0$$

for an arbitrary function $\psi(t)$ infinitely differentiable with respect to $t$, and with values in the space $H$ equal to zero for $t > T$ (the value $T < \infty$ is its own for each function $\psi$).

Using the existence theorems proved in [2], the definition of the generalized solution of Problem A may be given the form of the corresponding definition from [2]. Therefore, the following theorem is valid.

Theorem 3.1. Let all $\mathbf{b}_s$ from the initial conditions (1.2) belong to the space $\mathbf{H}$, and $C(\partial_t)\mathbf{F} \in L^2(R_+, \mathbf{L}^{6/5}(\Omega))$. Furthermore, let all the roots $p_s$ of the polynomial $P_0(p, \alpha, \beta)$ lie in the domain $\operatorname{Re} p < 0$ and $d_0(\alpha) \neq 0$ for all $\alpha \in [0, 1]$, $\beta \in [0, \beta_0]$.

Then there exists a unique generalized solution $\mathbf{u}(t)$ of Problem A belonging to the space $P_k(0, \mathbf{H}) \cap P_r(0, \mathbf{L}^2(\Omega))$, where $k = r = m_B$ if $m_B \geqslant m_A$, $m_B \geqslant m_C + 1$ and $k + 1 = r = m_B$ if $m_B \geqslant m_A$, $m_B = m_c > 0$ and

$$\inf [c_1 d_0(\alpha) - c_0 d_1(\alpha)] = m_2 > 0 \quad \text{for } \alpha \in [0, 1]$$

Here $d_s(\alpha) = \beta_s(1 + \alpha) + \alpha\alpha_{s-j}$, $j = m_B - m_A$, $\alpha_s = 0$ for $s < 0$, and

$\alpha_s$, $\beta_s$ are coefficients of the polynomials $A\ (p)$ and $B\ (p)$, respectively.

The existence of a unique "complete" generalized solution of Problem A follows from Theorem 3.1 by virtue of the estimates of the solution. A direct corollary to Theorem 3.1 is:

Theorem 3.2. Let all the conditions of Theorem 3.1 be satisfied. Then

$$\int\limits_M^\infty \Big\{\sum_{s=0}^k \|\partial_t^s \mathbf{a}\|_{\mathbf{H}}^2 + \sum_{s=0}^r \|\partial_t^s \mathbf{a}\|_{\mathbf{L}^2(\Omega)}^2\Big\}\, dt \to 0 \quad \text{for} \quad M \to \infty$$

As a corollary from Theorem 3.2 and the inequality [4],

$$\|\partial_t^l \varphi\|_H^2 \leqslant m_7 \int\limits_M^\infty \sum_{s=0}^g \|\partial_t^s \varphi\|_H^2\, dt, \quad l = 0, \ldots, g-1 \geqslant 0$$

with a constant $m_7$ independent of $M$, $\varphi\ (t)$, there results that as $t \to \infty$

$$\|\partial_t^l \mathbf{a}\|_{\mathbf{L}^2(\Omega)} \to 0, \quad l = 0, \ldots, r-1$$

$$\|\partial_t^s \mathbf{a}\|_{\mathbf{H}} \to 0, \quad s = 0, \ldots, k-1, \qquad \text{if } k > 0$$

uniformly in $t$.

It hence follows that the conditions of Theorem 3.1 are sufficient conditions for compliance of the partial principle Y.

Note 3.1. All the results obtained above are carried over directly to the case of a mixed initial-boundary value problem of linear viscoelasticity (part of the body boundary is rigidly fixed, and a surface load $\mathbf{f}_1$ acts on its other part). Here only changes which must hence be inserted in the conditions of the corresponding theorems will be noted.

The space $\mathbf{H}$ is replaced by $\mathbf{H}_1$, where $\mathbf{H}_1$ is the closure of vector-functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, continuously differentiable in $\Omega$ which satisfy homogeneous geometric conditions for supporting the body (such as the Korn inequality [8] is satisfied for $\varphi$) in the norm induced by the scalar product

$$(\varphi\cdot\psi)_{\mathbf{H}} = \frac{1}{4}\int\limits_\Omega (\varphi_{k,l} + \varphi_{l,k})(\overline{\psi}_{k,l} + \overline{\psi}_{l,k})\, d\Omega$$

The polynomial $P_0\ (p,\ \alpha,\ \beta)$ and $d_s\ (\alpha)$ are replaced by

$$P_1\ (p,\ \alpha,\ \beta) = 2B\ (p) + \alpha A\ (p) + \beta p^2 C\ (p)$$

$$d_{s1}\ (\alpha) = 2\beta_s + \alpha\alpha_{s-j}, \qquad j = m_B - m_A$$

The range of variation of the parameter $\alpha$ is $[0,\ 1]$ to $[0,\ 3]$; this is a result of the known inequality [7]

$$\int\limits_\Omega |\operatorname{div} \varphi|^2\, d\Omega \leqslant 3\|\varphi\|_{\mathbf{H}_1}^2$$

Moreover, it is necessary to impose everywhere the condition

$$C\ (\partial_t)\ \mathbf{f}_1 \in L^2\ (R_+,\ \mathbf{L}^{4/3}\ (\Gamma_1))$$

where $\Gamma_1$ is the part of the body boundary on which the load $\mathbf{f}_1$ acts.

Note 3.2. When the body boundary is not clamped, or the support allows displacement of the body as a rigid whole, corresponding theorems also hold in a formulation which agrees with the formulation of the theorem from Note 3.1. Not for the whole displacement vector $\mathbf{a}$, but for its "deformation" part $\mathbf{a}_1$, which is extracted as follows: follows:

$$\mathbf{a} = \mathbf{a}_1 + \sum_{s=1}^{m} \chi_s(t)\boldsymbol{\psi}_s, \quad (\mathbf{a}_1 \cdot \boldsymbol{\psi}_s)_{H_1} = 0, \quad s = 1, \ldots, m$$

where $\boldsymbol{\psi}_s$ is the basis of the rigid displacement vectors ($m = 6$ in the case of an unfixed boundary).

Note 3.3. For each specific problem of the linear viscoelasticity problems posed, the domain of variation of the parameter $\beta$ is bounded. For partial principle Y to be satisfied simultaneously for all such specific problems, it is necessary to require that all roots of the appropriate polynomials $P_s(p, \alpha, \beta)$ lie in the left half-plane of the complex variable $p$ for all $\alpha \in [0, \alpha_s^\circ]$, $\beta \in R_+$.

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# STATE OF STRESS IN A FLAT CIRCULAR RING WITH A CRACK

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The stress distribution in a circular isotropic ring with a crack on part of the concentric circle is investigated. A system of functional equations governing the coefficients of the complex Fourier series expansion of the stresses acting on the circle on which the crack is located is obtained. The solution of the mentioned system of equations is obtained by using a factorization method, which permitted reduction of the initial system of equations to two coupled infinite systems of algebraic equations. The possibility of using the method